Computational Complexity of Strong Admissibility for Abstract Dialectical Frameworks

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Abstract

Abstract dialectical frameworks (ADFs) have been introduced as a formalism for modeling and evaluating argumentation allowing general logical satisfaction conditions. Different criteria used to settle the acceptance of arguments are called semantics. Semantics of ADFs have so far mainly been defined based on the concept of admissibility. Recently, the notion of strong admissibility has been introduced for ADFs. In the current work we study the computational complexity of the following reasoning tasks under strong admissibility semantics. We address 1. the credulous/skeptical decision problem; 2. the verification problem; 3. the strong justification problem; and 4. the problem of finding a smallest witness of strong justification of a queried argument.

1 Introduction

Interest and attention in argumentation theory has been increasing among artificial intelligence researchers (Bench-Capon and Dunne 2007). Applications of argumentation theory are based on a variety of argumentation formalisms and methods of evaluating arguments (Atkinson et al. 2017; Baroni et al. 2018; van Eemeren et al. 2014). Dung's abstract argumentation frameworks (Dung 1995) (AFs for short) have received notable attention, also thanks to their simple syntax that can model and evaluate a number of non-monotonic reasoning tasks. Semantics of AFs single out coherent subsets of arguments that fit together, according to specific criteria (Baroni, Caminada, and Giacomin 2011).

AFs model individual attack relations among arguments. Abstract dialectical frameworks (ADFs) are expressive generalizations of AFs in which the logical relations among arguments can be represented. ADFs were first introduced in (Brewka and Woltran 2010), and were further refined in (Brewka et al. 2013; Brewka et al. 2017; Brewka et al. 2018).

Often a new semantics is a refinement of an already existing one by introducing further restrictions on the set of accepted arguments or possible attackers. One of the main types of semantics of AFs is the grounded semantics. Its characteristics include that 1. each AF has a unique grounded extension; 2. the grounded extension collects all the arguments about which no one doubts their acceptance; 3. the grounded extension is often a subset of the set of extensions of other types of AF semantics. Thus, it is important to investigate whether an argument belongs to the grounded extension of a given AF. The notion of strong admissibility is introduced for AFs to answer the query 'Why does an argument belong to the grounded extension?'.

While the grounded extension collects all the arguments of a given AF that can be accepted without any doubt, a strongly admissible extension provides a (minimal) justification why specific arguments can be accepted without any doubt, i.e. belong to the grounded extension. Thus, the strong admissibility semantics can be the basis for an algorithm that can be used not only for answering the credulous decision problem but also for human-machine interaction that requires an explainable outcome (cf. (Caminada and Uebis 2020; Booth, Caminada, and Marshall 2018)).

In AFs, the concept of strong admissibility semantics has first been defined in the work of Baroni and Giacomin (2007), and later in (Caminada 2014). Furthermore, in (2019), Caminada and Dunne presented a labelling account of strong admissibility to answer the decision problems of AFs under grounded semantics. Moreover, Caminada showed in (2018; 2014) that strong admissibility plays a role in discussion games for AFs under grounded semantics. In addition, the computational complexity of strong admissibility of AFs has been analyzed (Caminada and Dunne 2020; Dvořák and Wallner 2020).

Because of the specific structure of ADFs, the definition of strong admissibility semantics of AFs cannot be directly reused in ADFs. Thus the concept of strong admissibility for ADFs has been introduced (Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021a). This concept fulfils properties that are related to those of the strong admissibility semantics for AFs, as follows:

1. Strong admissibility is defined in terms of strongly justified arguments. 2. Strongly justified arguments are recursively reconstructed from their strongly justified parents. 3. Each ADF has at least one strongly admissible interpretation. 4. The set of strongly admissible interpretations of ADFs forms a lattice with as least element the trivial interpretation and as maximum element the grounded interpretation. 5. The strong admissibility semantics can be used to answer whether an argument is justifiable under grounded semantics. 6. The strong admissibility semantics of ADFs is different from the admissible, conflict-free, complete and grounded semantics of ADFs. 7. The strong admissibility

semantics for ADFs is a proper generalization of the strong admissibility semantics for AFs.

Whereas several fundamental properties of strong admissibility semantics for ADFs have been established, the computational complexity under strong admissibility semantics has not been studied. This work closes this gap by studying the complexity of the central reasoning tasks under the strong admissibility semantics of ADFs, as follows. 1. The credulous decision problem, i.e., whether there exists a strongly admissible interpretation that satisfies the queried argument, is coNP-complete. 2. The skeptical decision problem, i.e., whether all strongly admissible interpretations satisfy a queried argument, is trivial. 3. The verification problem, i.e., whether a given interpretation is a strongly admissible interpretation of an ADF, is coNP-complete. 4. The strong justification problem for an argument in an interpretation, i.e., whether an argument is strongly justified in an interpretation, is coNP-complete. 5. The problem of finding a small witness of strong justification of an argument, i.e, whether there exists a strongly admissible interpretation that satisfies a queried argument and is smaller than a given bound, is Σ_2^{P}-complete.

2 Formal Background

We recall the basics of AFs (Dung 1995) and ADFs (Brewka et al. 2018). Also we recall the definition of strong admissibility for ADFs and an associated algorithm, presented in (Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021b).

2.1 Abstract Argumentation Frameworks

We start the preliminaries to our work by recalling the basic notion of Dung's abstract argumentation frameworks (AFs). Subsequently, we present the extension form of strong admissibility semantics of AFs (Baroni and Giacomin 2007).

Definition 1. (Dung 1995) An abstract argumentation framework (AF) is a pair (A, R) in which A is a set of arguments and $R \subseteq A \times A$ is a binary relation representing attacks among arguments.

Let $F = (A, R)$ be a an AF. For each $a, b \in A$, the relation $(a, b) \in R$ is used to represent that a is an argument attacking the argument b. An argument $a \in A$ is, on the other hand, defended by a set $S \subseteq A$ of arguments (alternatively, the argument is acceptable with respect to S) (in F) if for each argument $c \in A$, it holds that if $(c, a) \in R$ then there is an $s \in S$ such that $(s, c) \in R$ (s is called a defender of a).

Different semantics of AFs present which sets of arguments in an AF can be jointly accepted (see the overview (Baroni, Caminada, and Giacomin 2011)). Set $S \subseteq A$ is called a *conflict-free* extension (in F) if there are no $a, b \in S$ s.t. $(a, b) \in R$. The *characteristic function* F : $2^A \mapsto 2^A$ is defined as $F(S) = \{a \mid a \text{ is defended by } S\}$. Set $S \subseteq A$ is called an *admissible* extension (in F) if $S \subseteq F(S)$. Further, set $S \subseteq A$ is a *grounded* extension of an AF if S is the $\subseteq$-least fixed point of F.

Definition 2. (Baroni and Giacomin 2007) Given an argumentation framework $F = (A, R)$, $a \in A$ and $S \subseteq A$, it is said that a is *strongly defended* by S if and only if each attacker $c \in A$ of a is attacked by some $s \in S \setminus \{a\}$ such that s is strongly defended by $S \setminus \{a\}$.

Example 1. Let $F = (\{a, b, c\}, \{(a, b), (b, c)\})$ be an AF. Argument a is strongly defended by $S = \emptyset$, since a is not attacked by any argument. Also, argument c is strongly defended by set $S = \{a, c\}$, since the attacker of c, namely b is attacked by $a \in S \setminus \{c\}$ and a itself is strongly defended.

Definition 3. Given an AF (A, R) and set $S \subseteq A$, it is said that S is a *strongly admissible* extension of S if every $s \in S$ is strongly defended by S.

In Example 1, sets $S_1 = \emptyset$, $S_2 = \{a\}$, and $S_3 = \{a, c\}$ are strongly admissible extensions of F; all of them are subsets of the grounded extension of F. However, set $S' = \{c\}$ is not a strongly admissible extension of F, since $c \in S'$ is not strongly defended by $S' \setminus \{c\}$. Because argument c is attacked by b, however, no argument in $S' \setminus \{c\}$ attacks b.

2.2 Abstract Dialectical Frameworks

We summarize key concepts of abstract dialectical frameworks (Brewka and Woltran 2010; Brewka et al. 2018).

Definition 4. An abstract dialectical framework (ADF) is a tuple $D = (A, L, C)$ where:

- A is a finite set of arguments (statements, positions);
- $L \subseteq A \times A$ is a set of links among arguments;
- $C = \{\varphi_a\}_{a \in A}$ is a collection of propositional formulas over arguments, called acceptance conditions.

An ADF can be represented by a graph in which nodes indicate arguments and links show the relation among arguments. Each argument a in an ADF is labelled by a propositional formula, called acceptance condition, φ_a over $par(a)$ such that, $par(a) = \{b \mid (b, a) \in L\}$. The acceptance condition of each argument clarifies under which condition the argument can be accepted. An argument a is called an *initial argument* if $par(a) = \{\}$.

A *three-valued interpretation* v (for D) is a function $v : A \mapsto \{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$, that maps arguments to one of the three truth values true ($\mathbf{t}$), false ($\mathbf{f}$), or undecided ($\mathbf{u}$). Interpretation v is called *trivial*, and v is denoted by $v_{\mathbf{u}}$, if $v(a) = \mathbf{u}$ for each $a \in A$. Further, v is called a two-valued interpretation if for each $a \in A$ either $v(a) = \mathbf{t}$ or $v(a) = \mathbf{f}$.

Truth values can be ordered via the information ordering relation $<_i$ given by $\mathbf{u} <_i \mathbf{t}$ and $\mathbf{u} <_i \mathbf{f}$ and no other pair of truth values are related by $<_i$. Relation $\leq_i$ is the reflexive closure of $<_i$. The pair $(\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}, \leq_i)$ is a complete meet-semilattice with the meet operator $\sqcap_i$, such that $\mathbf{t} \sqcap_i \mathbf{t} = \mathbf{t}$, $\mathbf{f} \sqcap_i \mathbf{f} = \mathbf{f}$, and returns $\mathbf{u}$ otherwise. The meet of two interpretations v and w is then defined as $(v \sqcap_i w)(a) = v(a) \sqcap_i w(a)$ for all $a \in A$.

It is said that an interpretation v is an *extension* of another interpretation w, if $w(a) \leq_i v(a)$ for each $a \in A$, denoted by $w \leq_i v$. Further, if $v \leq_i w$ and $w \leq_i v$, then v and w are equivalent, denoted by $v \sim_i w$.

For reasons of brevity, we will shorten the notion of three-valued interpretation $v = \{a_1 \mapsto t_1, \ldots, a_m \mapsto t_m\}$ with arguments $a_1, \ldots, a_m$ and truth values $t_1, \ldots, t_m$ as follows: $v = \{a_i \mid v(a_i) = \mathbf{t}\} \cup \{\neg a_i \mid v(a_i) = \mathbf{f}\}$. For instance, $v = \{a \mapsto \mathbf{f}, b \mapsto \mathbf{t}\} = \{\neg a, b\}$.

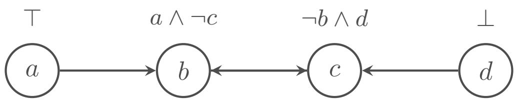

Figure 1: ADF of Examples 2 and 3

Given an interpretation v (for D), the partial valuation of φ_a by v is $v(\varphi_a) = \varphi_a^v = \varphi_a[b/\top : v(b) = \mathbf{t}][b/\bot : v(b) = \mathbf{f}]$, for $b \in par(a)$. Semantics for ADFs can be defined via the *characteristic operator* Γ_D, presented in Definition 5.

Definition 5. Let D be an ADF and let v be an interpretation of D. Applying Γ_D on v leads to v' such that for each $a \in A$, v' is as follows:

$$v'(a) = \begin{cases} \mathbf{t} & \text{if } \varphi_a^v \text{ is irrefutable (i.e., } \varphi_a^v \text{ is a tautology)}, \\ \mathbf{f} & \text{if } \varphi_a^v \text{ is unsatisfiable}, \\ \mathbf{u} & \text{otherwise.} \end{cases}$$

Most types of semantics for ADFs are based on the concept of admissibility. An interpretation v for a given ADF F is called *admissible* iff $v \leq_i \Gamma_F(v)$; it is *preferred* iff v is $\leq_i$-maximal admissible; it the *grounded* interpretation of D iff v is the least fixed point of Γ_D. The set of all σ interpretations for an ADF D is denoted by $\sigma(D)$, where $\sigma \in \{adm, grd, prf\}$ abbreviates the different semantics in the obvious manner.

Example 2. An example of an ADF $D = (S, L, C)$ is shown in Figure 1. To each argument a propositional formula is associated, namely, the acceptance condition of the argument. For instance, the acceptance condition of c, namely $\varphi_c : \neg b \wedge d$, states that c can be accepted in an interpretation in which b is denied and d is accepted.

The interpretation $v_1 = \{a, \neg c, \neg d\}$ is an admissible interpretation, since $\Gamma_D(v_1) = \{a, b, \neg c, \neg d\}$ and $v_1 \leq_i \Gamma_D(v_1)$. Furthermore, $v_2 = \{a, b, \neg c, \neg d\}$ is a unique grounded interpretation and a preferred interpretation in D.

The notions of an argument being acceptable or deniable in an interpretation are defined as follows.

Definition 6. Let $D = (A, L, C)$ be an ADF and let v be an interpretation of D.

- An argument $a \in A$ is called *acceptable* with respect to v if φ_a^v is irrefutable.
- An argument $a \in A$ is called *deniable* with respect to v if φ_a^v is unsatisfiable.

We say that an argument is justified with respect to v if it is either acceptable or deniable with respect to v.

We redefine two decision problems of ADFs in Definition 7.

Definition 7. Let $D = (A, L, C)$ be an ADF, let σ be semantics of ADFs, i.e., $\sigma \in \{adm, prf, grd, cf\}$, and let a be an argument of A.

- a is *credulously acceptable (deniable)* under σ if there exists an interpretation v with $v \in \sigma(D)$ in which $v(a) = \mathbf{t}$ ($v(a) = \mathbf{f}$, respectively), denoted by $Cred_\sigma$.
- a is *skeptically acceptable (deniable)* under σ if for each v with $v \in \sigma(D)$ it holds that $v(a) = \mathbf{t}$ ($v(a) = \mathbf{f}$, respectively).

2.3 The Strong Admissibility Semantics for ADFs

In this section, we rephrase the concept of strong admissibility semantics for ADFs from (Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021a), which is defined based on the notion of strongly justifiable arguments (i.e., strongly acceptable/deniable arguments). Below, the interpretation $v_{|P}$ is equal to $v(p)$ for any $p \in P$, and assigns all arguments that do not belong to P to $\mathbf{u}$, i.e., $v_{|P} = v_{\mathbf{u}}|^{p \in P}_{v(p)}$.

Definition 8. Let $D = (A, L, C)$ be an ADF and let v be an interpretation of D. Argument a is a *strongly justified* argument in interpretation v with respect to set E if one of the following conditions holds:

- $v(a) = \mathbf{t}$ and there exists a subset of parents of a excluding E, namely $P \subseteq par(a) \setminus E$ such that (a) a is acceptable with respect to $v_{|P}$ and (b) all $p \in P$ are strongly justified in v with respect to set $E \cup \{p\}$.
- $v(a) = \mathbf{f}$ and there exists a subset of parents of a excluding E, namely $P \subseteq par(a) \setminus E$ such that (a) a is deniable with respect to $v_{|P}$ and (b) all $p \in P$ are strongly justified in v with respect to set $E \cup \{p\}$.

An argument a is *strongly acceptable*, resp. *strongly deniable*, in v if $v(a) = \mathbf{t}$, resp. $v(a) = \mathbf{f}$, and a is strongly justified in v with respect to set $\{a\}$. We further say that an argument is *strongly justified* in v if it is either strongly acceptable or deniable in v.

Note that in Definition 8, the set of parents of a can be the empty set, i.e., $P = \emptyset$. If the set of parents of an argument, is empty, then $v_{|P} = v_{\mathbf{u}}$. In this case, a is strongly acceptable/deniable in v if $\varphi_a^{v_{\mathbf{u}}}$ is irrefutable/unsatisfiable, respectively. We say that a *is not strongly justified in an interpretation* v if there is no such a set of parents of a that satisfies the conditions of Definition 8 for a. The notion of strongly justified arguments in a given interpretation is presented in Example 3.

Example 3. Let $D = (\{a, b, c, d\}, \{\varphi_a : \top, \varphi_b : a \wedge \neg c, \varphi_c : \neg b \wedge d, \varphi_d : \bot\})$ be the ADF depicted in Figure 1. Let $v = \{b, \neg c, \neg d\}$. We show that c and d are strongly justified in v and b is not strongly justified in v. Since $v(c) = v(d) = \mathbf{f}$, we show that c and d are strongly deniable in v. First, since $\varphi_d^{v_{\mathbf{u}}} \equiv \bot$, it holds that d is strongly deniable in v.

We show that c is strongly deniable in v with respect to $E = \{c\}$. we choose the subset of parents of c excluding c equal to $P = \{d\}$. It is easy to check that $\varphi_c^{v_{|P}}$ is unsatisfiable, i.e., $\varphi_c^{v_{|P}} \equiv \varphi_c^{v_{|d}} \equiv \bot$. That is, c is deniable w.r.t. $v_{|d}$. Then, since $d \in P$, $v(d) = \mathbf{f}$ and d is strongly justified in v with respect to $E = \{c, d\}$, c is strongly deniable in v.

To show that b is not strongly justified in v, since $v(b) = \mathbf{t}$, we show that b is not strongly acceptable in v. Toward a contradiction, assume that b is strongly acceptable in v. Thus, we have to choose a set of parents of b, namely P that satisfies $\varphi_b^{v_{|P}} \equiv \top$. Let $P = par(b)$. Since $\varphi_b^{v_{|P}} \not\equiv \top$, there is not subset of $par(b)$ that satisfies the conditions of Definition 8 for b. Thus, b is not strongly acceptable in v.

In Example 3, if we choose a set of parents of c equal to $\{b\}$, then we cannot show that c is strongly deniable in interpretation v. The reason is that b is not strongly justified in v,

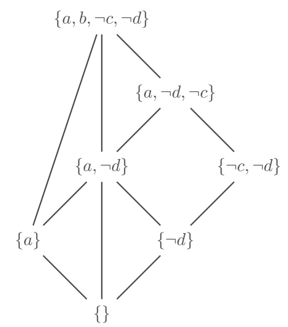

Figure 2: Complete lattice of the strongly admissible interpretations of the ADF of Example 3

as is presented in Example 3. This shows the importance of choosing a right set of parents that satisfies the conditions of Definition 8 for a queried argument. However, there exists an alternative method for checking whether an argument is strongly justified, presented in (Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021b), in which there is no need of indicating a set of parents of a queried argument.

Definition 9. Let $D = (A, L, C)$ be an ADF and let v be an interpretation of D. An interpretation v is a *strongly admissible* interpretation if for each a such that $v(a) = \mathbf{t}/\mathbf{f}$, it holds that a is a strongly justified argument in v.

To clarify the notion of strongly admissible interpretations of ADFs, we continue Example 3 in Example 4.

Example 4. Consider again the ADF of Example 3, i.e., $D = (\{a, b, c, d\}, \{\varphi_a : \top, \varphi_b : a \wedge \neg c, \varphi_c : \neg b \wedge d, \varphi_d : \bot\})$, depicted in Figure 1. Let $v = \{b, \neg c, \neg d\}$. As shown in Example 3, c and d are strongly justified in v. However, b is not strongly justified in v. Thus, v is not a strongly admissible interpretation of D. However, for instance, $v_1 = \{a\}$, $v_2 = \{\neg c, \neg d\}$ and $v_3 = \{a, b, \neg c, \neg d\}$ are strongly admissible interpretations of D. We show that b is strongly acceptable in v_3. To this end, let $P = \{a, c\}$ be a set of parents of b. First, it holds that $\varphi_b^{v_3|_P} \equiv \top$. Thus, the first condition is satisfied for b. We also have to check whether each parent of b is strongly justified in v_3. To this end, we show that a is strongly acceptable in v_3 and c is strongly deniable in v_3. The latter is obvious by the same method that was presented in Example 3 to show that c is strongly deniable in v. In addition, $\varphi_a^{v_\mathbf{u}} \equiv \top$, thus, a is strongly acceptable in v_3. Hence, b and a are strongly justified in v_3. Furthermore, v_3 is a unique grounded interpretation of D.

It is shown in (2021b) that the strongly admissible interpretations of D form a lattice with respect to the $\leq_i$-ordering, with the least element being $v_\mathbf{u}$ and the maximum element being the grounded interpretation of D. The set of strongly admissible interpretations of ADF D given in Example 3 form a lattice, depicted in Figure 2.

2.4 Algorithm for Strongly Admissible Interpretations of ADFs

In this section we review an existing method, presented in Section 5 of (Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021b), to answer the verification problem under strong admissibility semantics. To this end, we introduce $\Gamma_{D,v}$, a variant of the characteristic operator restricted to a given interpretation v.

Definition 10. Let D be an ADF and let v, w be interpretations of D. Let $\Gamma_{D,v}(w) = \Gamma_D(w) \sqcap_i v$, where $\Gamma^n_{D,v}(w) = \Gamma_{D,v}(\Gamma^{n-1}_{D,v}(w))$ for n with $n \geq 1$, and $\Gamma^0_{D,v}(w) = w$.

We next use the $\Gamma_{D,v}$ operator to recall observations on the sequence of interpretations generated by a least fixed-point iteration on $\Gamma_{D,v}$.

Lemma 1 ((Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021b))**.** *Let $D = (A, L, C)$ be a given ADF and let v be an interpretation of D. Let $\Gamma^n_{D,v}(v_\mathbf{u})$ be the set of interpretations constructed based on v, as in Definition 10. For each i it holds that;*

- $\Gamma^i_{D,v}(v_\mathbf{u}) \leq_i \Gamma^{i+1}_{D,v}(v_\mathbf{u})$*;*
- $\Gamma^i_{D,v}(v_\mathbf{u})$ *is a strongly admissible interpretation of D;*
- *if $\Gamma^i_{D,v}(v_\mathbf{u})(a) = \mathbf{t}/\mathbf{f}$, then a is strongly justifiable in $\Gamma^i_{D,v}(v_\mathbf{u})$.*

The sequence of interpretations $\Gamma^i_{D,v}(v_\mathbf{u})$ as defined in Definition 10 is named the sequence of strongly admissible interpretations constructed based on v in D.

Based on the above observations, one can characterise strongly admissible interpretations v as least fixed point of the corresponding operator $\Gamma_{D,v}$. That is, we can verify an interpretation by computing this sequence of strongly admissible interpretations.

Theorem 1 ((Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021b))**.** Let D be an ADF and let v be an interpretation of D. Let $\Gamma^i_{D,v}(v_\mathbf{u})$ (for $i \geq 0$) be the sequence of strongly admissible interpretations constructed based on v in D. The following conditions hold:

- there is an m with $m \geq 0$ s.t. $\Gamma^m_{D,v}(v_\mathbf{u}) \sim_i \Gamma^{m+1}_{D,v}(v_\mathbf{u})$;
- $v \in \mathit{sadm}(D)$ iff there exists an m s.t. $v \sim_i \Gamma^m_{D,v}(v_\mathbf{u})$.

Example 5 illustrates the role of Theorem 1 in the verification problem under the strong admissibility semantics.

Example 5. Consider again the ADF given in Example 3, i.e., $D = (\{a, b, c, d\}, \{\varphi_a : \top, \varphi_b : a \wedge \neg c, \varphi_c : \neg b \wedge d, \varphi_d : \bot\})$. Let $v = \{a, \neg c, \neg d\}$. We check whether $v \in \mathit{sadm}(D)$ based on the method presented in Theorem 1. The sequence of strongly admissible interpretations constructed based on v is as follows.
$v_1 = \Gamma_{D,v}(v_\mathbf{u}) = \{a, \neg d\} \sqcap_i \{a, \neg c, \neg d\} = \{a, \neg d\}$;
$v_2 = \Gamma^2_{D,v}(v_\mathbf{u}) = \{a, \neg c, \neg d\} \sqcap_i \{a, \neg c, \neg d\} = \{a, \neg c, \neg d\}$.
Since $v \sim_i \Gamma^2_{D,v}(v_\mathbf{u})$, it holds that $v \in \mathit{sadm}(D)$.

On the other hand, let $v' = \{a, b\}$. We show that $v' \notin \mathit{sadm}(D)$. The sequence of interpretations constructed based on v' is as follows:
$v_1 = \Gamma_D(v_\mathbf{u}) \sqcap_i v' = \{a, \neg d\} \sqcap_i \{a, b\} = \{a\}$;
$v_2 = \Gamma_D(v_1) \sqcap_i v' = \{a, \neg d\} \sqcap_i \{a, b\} = \{a\}$.
Thus, the sequence of interpretations constructed based on v' leads to $v_2 = \{a\}$, which is not equal to v', i.e., $v' \not\sim_i v_2$. Hence, v' is not a strongly admissible interpretation of D.

Based on the above results (Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021b) provides algorithms that decide (a) verification of a given strongly admissible interpretation and (b) whether an argument is strongly acceptable/deniable within a given interpretation that are based on an iterative fixed-point computation of an operator $\Gamma_{D,v}$. However, because testing whether an argument is acceptable in Γ_D is already NP/coNP-hard (Dvořák and Dunne 2018), these procedures are in P^{NP} and as we will show, both problems allow for algorithms of significantly lower complexity.

3 Computational Complexity

We analyse the complexity under strong admissibility semantics for (a) the standard reasoning tasks of ADFs (Dvořák and Dunne 2018) and (b) two problems specific to strong admissibility semantics, i.e., the small witness problem introduced for AFs in (Dvořák and Wallner 2020; Caminada and Dunne 2020) and the strong justification problem.

For a given ADF D we consider the following problems:

1. *The credulous decision problem*: whether an argument a is credulously justifiable with respect to the strong admissibility semantics of D. That is, if there exists a strongly admissible interpretation of D in which a is strongly justified. This reasoning task is denoted as $\mathit{Cred}_{\mathit{sadm}}(a \mapsto \mathbf{t}/\mathbf{f}, D)$ and is presented formally as follows:

$$\mathit{Cred}_{\mathit{sadm}}(a \mapsto \mathbf{t}/\mathbf{f}, D) = \begin{cases} \text{yes} & \text{if } \exists v \in \mathit{sadm}(D) \text{ s.t.} \\ & v(a) = \mathbf{t}/\mathbf{f}, \\ \text{no} & \text{otherwise} \end{cases}$$

2. *The skeptical decision problem*: whether an argument a is skeptically justified with respect to the strong admissibility semantics of D. That is, if a is strongly justified in all strongly admissible interpretations of D, denoted as $\mathit{Skept}_{\mathit{sadm}}(a \mapsto \mathbf{t}/\mathbf{f}, D)$, which is presented formally as follows:

$$\mathit{Skept}_{\mathit{sadm}}(a \mapsto \mathbf{t}/\mathbf{f}, D) = \begin{cases} \text{yes} & \text{if } \forall v \in \mathit{sadm}(D): \\ & v(a) = \mathbf{t}/\mathbf{f} \text{ holds}, \\ \text{no} & \text{otherwise} \end{cases}$$

3. *The verification problem*: whether a given interpretation v is a strongly admissible interpretation of D, denoted by $\mathit{Ver}_{\mathit{sadm}}(v, D)$, which is presented formally as follows:

$$\mathit{Ver}_{\mathit{sadm}}(v, D) = \begin{cases} \text{yes} & \text{if } v \in \mathit{sadm}(D), \\ \text{no} & \text{otherwise} \end{cases}$$

4. *The strong justification problem:* The problem whether a given argument a is strongly justified in a given interpretation v is denoted as $\mathit{StrJust}(a \mapsto \mathbf{t}/\mathbf{f}, v, D)$, which is presented formally as follows:

$$\mathit{StrJust}(a \mapsto \mathbf{t}/\mathbf{f}, v, D) = \begin{cases} \text{yes} & \text{if } a \text{ is strongly} \\ & \text{justified in } v, \\ \text{no} & \text{otherwise} \end{cases}$$

5. *The small witness problem:* We are interested in computing a strongly admissible interpretation that has the least information of the ancestors of a given argument, namely a, where $v(a) = \mathbf{t}/\mathbf{f}$. The decision version of this problem is the k-Witness problem, denoted by $k\text{-}\mathit{Witness}_{\mathit{sadm}}$, indicating whether a given argument is strongly justified in at least one v such that $v \in \mathit{sadm}(D)$ and $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| \leq k$. Note that k is part of the input of this problem. This decision problem is presented formally as follows:

$$k\text{-}\mathit{Witness}_{\mathit{sadm}}(a \mapsto \mathbf{t}/\mathbf{f}, D) = \begin{cases} \text{yes} & \text{if } \exists v \in \mathit{sadm}(D) \\ & \text{s.t. } v(a) = \mathbf{t}/\mathbf{f} \\ & \& |v^{\mathbf{t}} \cup v^{\mathbf{f}}| \leq k, \\ \text{no} & \text{otherwise} \end{cases}$$

3.1 The Credulous/Skeptical Decision Problems

In this section we study the credulous/skeptical problem under the strong admissibility semantics for ADFs. That is, we show the complexity of deciding whether an argument in question is credulously/skeptically justifiable in at least one/all strongly admissible interpretation(s) of a given ADF.

We show that $\mathit{Cred}_{\mathit{sadm}}$ is coNP-complete and $\mathit{Skept}_{\mathit{sadm}}$ is trivial. To this end, we use the fact that the set of strongly admissible interpretations of a given ADF D forms a lattice with respect to the $\leq_i$-ordering, with the maximum element being $\mathit{grd}(D)$. Thus, any strongly admissible interpretation of D has at most an amount of information equal to $\mathit{grd}(D)$. Thus, answering the credulous decision problem under the strong admissibility semantics coincides with answering the credulous decision problem under the grounded semantics.

Theorem 2. $\mathit{Cred}_{\mathit{sadm}}$ is coNP-complete.

Proof. We have that $\mathit{Cred}_{\mathit{sadm}}(a \mapsto \mathbf{t}/\mathbf{f}, D) = \mathit{Cred}_{\mathit{grd}}(a \mapsto \mathbf{t}/\mathbf{f}, D)$ and the latter has been shown to be coNP-complete in (Wallner 2014, Proposition 4.1.3.). □

Concerning skeptical acceptance, notice that the trivial interpretation is the least strongly admissible interpretation in each ADF. Thus, $\mathit{Skept}_{\mathit{sadm}}(a \mapsto \mathbf{t}/\mathbf{f}, D)$ is trivially *no*.

Theorem 3. $\mathit{Skept}_{\mathit{sadm}}$ is a trivial problem.

3.2 The Verification Problem

In this section, we settle the complexity of $\mathit{Ver}_{\mathit{sadm}}(v, D)$, i.e., of deciding whether a given interpretation v is a strongly admissible interpretation of an ADF D. We have seen in Section 2.4 that this problem can be solved in P^{NP}.

We first sketch a simple translation-based approach that reduces the verification problem of strongly admissible semantics to the verification problem of grounded semantics. In order to reduce $\mathit{Ver}_{\mathit{sadm}}(v, D)$ to $\mathit{Ver}_{\mathit{grd}}(v, D')$, we modify the acceptance conditions φ_a of D to $\varphi'_a = \neg a$ if $v(a) = \mathbf{u}$ and $\varphi'_a = \varphi_a$ otherwise. We then have that $v \in \mathit{sadm}(D)$ iff $v \in \mathit{grd}(D)$, so that we can use the DP procedure for $\mathit{Ver}_{\mathit{grd}}(v, D')$ (Wallner 2014, Theorem 4.1.4). This gives a DP procedure. However, as we will discuss next, $\mathit{Ver}_{\mathit{sadm}}(v, D)$ can be solved within coNP.

Intuitively, since the grounded interpretation is the maximum element of the lattice of strongly admissible interpretations and the credulous decision problem under grounded semantics is coNP-complete, it seems that the verification problem under the strong admissibility semantics has to be coNP-complete. However, having the positive answer for $Cred_{grd}(a \mapsto \mathbf{t}/\mathbf{f}, D)$ for each a with $v(a) = \mathbf{t}/\mathbf{f}$ does not lead to the positive answer of $Ver_{sadm}(v, D)$. This is because $v \leq_i grd(D)$ does not imply that v is a strongly admissible interpretation of D (see Example 6 below).

Example 6. Let $D = (\{a, b\}, \{\varphi_a : \top, \varphi_b : a \vee b\})$. The grounded interpretation of D is $\{a \mapsto \mathbf{t}, b \mapsto \mathbf{t}\}$. Furthermore, the interpretation $v = \{a \mapsto \mathbf{u}, b \mapsto \mathbf{t}\}$ is an admissible interpretation of D such that $v \leq_i grd(D)$. However, v is not a strongly admissible interpretation of D. As we know, the answer of $Cred_{grd}(b \mapsto \mathbf{t}, D)$ is *yes*, but b is not strongly acceptable in v. Thus, v is not a strongly admissible interpretation of D, i.e., the answer to $Ver_{sadm}(v, D)$ is *no*.

To show that Ver_{sadm} is coNP-complete, we modify and combine both the fixed-point iteration from Section 2.4 and the grounded algorithm from (Wallner 2014). To this end, we need some auxiliary results that are shown in Lemmas 2 and 3.

Lemma 2. *Given an ADF D with n arguments, the following statements are equivalent:*

1. *v is a strongly admissible interpretation of D;*
2. *$v = \Gamma^n_{D,v}(v_\mathbf{u})$;*
3. *for each $w \leq_i v$, it holds that $v = \Gamma^n_{D,v}(w)$.*

Proof.
- $1 \leftrightarrow 2$: by Theorem 1.
- $2 \mapsto 3$: Assume that $v = \Gamma^n_{D,v}(v_\mathbf{u})$ and that $w \leq_i v$. We show that $v = \Gamma^n_{D,v}(w)$. Since $v_\mathbf{u} \leq_i w \leq_i v$, and Γ_D is monotonic and thus also $\Gamma_{D,v}$ monotonic, we have $\Gamma^n_{D,v}(v_\mathbf{u}) \leq_i \Gamma^n_{D,v}(w) \leq_i \Gamma^n_{D,v}(v)$. Now using that $v = \Gamma^n_{D,v}(v_\mathbf{u})$, we obtain $v \leq_i \Gamma^n_{D,v}(w) \leq_i \Gamma^{2n}_{D,v}(v_\mathbf{u})$. Because $\Gamma_{D,v}$ is a monotonic operator, the fixed-point is reached after at most n iterations and thus $\Gamma^{2n}_{D,v}(v_\mathbf{u}) = \Gamma^n_{D,v}(v_\mathbf{u}) = v$. Hence, $\Gamma^n_{D,v}(w) = v$.
- $3 \mapsto 2$: Assume that for each $w \leq_i v$ it holds that $v \sim_i \Gamma^n_{D,v}(w)$. Thus, since $v_\mathbf{u} \leq_i v$, it holds that $v \sim_i \Gamma^n_{D,v}(v_\mathbf{u})$.

□

In the following, let $v^* = v^\mathbf{t} \cup v^\mathbf{f}$. The notions of completion of an interpretation and model are presented in Definition 11, used in Lemma 3.

Definition 11. Let w be an interpretation. We define the *completion* of w as the set of all two-valued extensions of w, denoted by $[w]_2$ where: $[w]_2 = \{u \mid w \leq_i u$ *and u is a two-valued interpretation*$\}$.

Furthermore, a two-valued interpretation u is said to be a *model* of formula φ, if $u(\varphi) = \mathbf{t}$, denoted by $u \models \varphi$.

Lemma 3. *Let D be an ADF and let v be an interpretation of D. $v \notin sadm(D)$ if and only if there exists an interpretation w of D that satisfies all the following conditions:*

1. *$w <_i v$;*
2. *For each $a \in w^\mathbf{u} \cap v^\mathbf{t}$ there exists $u_a \in [w]_2$ s.t. $u_a \not\models \varphi_a$;*
3. *For each $a \in w^\mathbf{u} \cap v^\mathbf{f}$ there exists $u_a \in [w]_2$ s.t. $u_a \models \varphi_a$.*

Proof. $\Leftarrow$: Assume that v and w are interpretations of D that satisfy all of the items 1, 2, 3 presented in the lemma. We show that $v \notin sadm(D)$. Toward a contradiction assume that $v \in sadm(v)$. Let a be an argument such that $a \in w^\mathbf{u} \cap v^\mathbf{t}$, thus, since w satisfies the conditions of the lemma, it holds that there exists $u_a \in [w]_2$ such that $u_a \not\models \varphi_a$, i.e., $u_a(a) = \mathbf{f}$. Furthermore, since $v(a) = \mathbf{t}$ and $v \in sadm(D)$, for any $j \in [v]_2$ it holds that $j \models \varphi_a$. Since $w <_i v$, it holds that $j \in [w]_2$, i.e., $\Gamma_D(w)(a) = \mathbf{u}$. The proof method for the case that $a \in w^\mathbf{u} \cap v^\mathbf{f}$ is similar, i.e., if $a \in w^\mathbf{u} \cap (v^\mathbf{t} \cup v^\mathbf{f})$, then $\Gamma_D(w)(a) = \mathbf{u}$. Thus, for $a \in w^\mathbf{u} \cap v^*$ we have $\Gamma_{D,v}(w)(a) = (\Gamma_D(w) \sqcap v)(a) = \mathbf{u}$. In other words, $\Gamma_{D,v}(w)(a) \leq_i w$ and thus, by the monotonicity of $\Gamma_{D,v}(w)$ also $\Gamma^n_{D,v}(w)(a) \leq_i w <_i v$. Thus, since $\Gamma^n_{D,v}(w) \not\sim_i v$ the third item of Lemma 2 does not hold for w with $w <_i v$. Thus, $v \notin sadm(D)$.
$\Rightarrow$: Assume that $v \notin sadm(D)$. That is, for the fixed point $w = \Gamma^n_{D,v}(v_\mathbf{u})$ we have $w <_i v$. Consider $a \in w^\mathbf{u} \cap v^\mathbf{t}$. Because w is a fixed point, we have that $\Gamma_{D,v}(w)(a) \neq \mathbf{t}$ and thus $\Gamma_D(w) \neq \mathbf{t}$. That is, there is a $u_a \in [w]_2$ such that $u_a \not\models \varphi_a$. Similar reasoning applies to $a \in w^\mathbf{u} \cap v^\mathbf{f}$. □

Lemma 4 shows that the verification problem is a coNP-problem, and Lemma 5 shows the hardness of this problem.

Lemma 4. *Ver_{sadm} is a coNP-problem for ADFs.*

Proof. Let D be an ADF and let v be an interpretation of D. For membership, consider the co-problem. By Lemma 3, if there exists an interpretation of w that satisfies the condition of Lemma 3, then v is not a strongly admissible interpretation of D. Thus, guess an interpretation w, together with an interpretation $u_a \in [w]_2$ for each $a \in v^*$, and check whether they satisfy the conditions of Lemma 3. Note that since $w <_i v$ we have to check the second and the third items of Lemma 3 a total of $|v^* \setminus w^\mathbf{u}|$ number of times. That is, this checking has to be done at most $|v^*|$ number of times, when w is the trivial interpretation. Thus, this checking step is linear in the size of v^*. Therefore, the procedure of guessing of w and checking if it satisfies 1, 2, 3 of Lemma 3 is an NP-problem. Thus, if a w satisfies the items of Lemma 3, then the answer to $Ver_{sadm}(v, D)$ is *no*. Otherwise, if we check all interpretations w such that $w <_i v$ and none of them satisfies the conditions of Lemma 3, then the answer to $Ver_{sadm}(v, D)$ is *yes*. Thus, $Ver_{sadm}(v, D)$ is a coNP-problem. □

Lemma 5. *Ver_{sadm} is coNP-hard for ADFs.*

Proof. For hardness of Ver_{sadm}, we consider the standard propositional logic problem of VALIDITY. Let ψ be an arbitrary Boolean formula and let $X = atom(\psi)$ be the set of atoms in ψ. Let a be a new atom, i.e., $a \notin X$. Construct ADF $D = (\{X \cup \{a\}\}, L, C)$ where $\varphi_x : x$ for each $x \in X$ and $\varphi_a : \psi$. We show that ψ is valid if and only if $v = v_\mathbf{u}|^a_\mathbf{t}$ is a strongly admissible interpretation of D. An illustration of the reduction for the formula $\psi = \neg b \vee b$ to the ADF $D = (\{a, b\}, L, \varphi_a : \psi, \varphi_b : b)$ is shown in Figure 3.

Assume that ψ is a valid formula. We show that v is the grounded interpretation of D. By the acceptance condition of each x, for $x \in X$ it is clear that x is assigned to $\mathbf{u}$ in the grounded interpretation of D. Further, since ψ is a valid

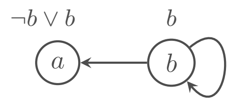

Figure 3: Reduction used in Lemma 5 and 9, for $\psi = \neg b \vee b$.

formula, it holds that ${\varphi_a}^{v_\mathbf{u}} \equiv \top$. Thus, the interpretation $v = v_\mathbf{u}|^a_\mathbf{t}$ is the grounded interpretation of D. Hence, $v \in$ *sadm*(D).

On the other hand, assume that ψ is not valid. Then there exists a two-valued interpretation v of $atom(\psi)$ such that $v \not\models \psi$. This implies that $a \mapsto \mathbf{t}$ does not belong to the grounded interpretation of D. Since the grounded interpretation of D is the maximum element of the lattice of strongly admissible interpretations, it holds that a is not strongly acceptable in any strongly admissible interpretation of D, that is, $v \notin$ *sadm*(D). □

Theorem 4 is a direct result of Lemmas 4–5.

Theorem 4. *Ver$_{sadm}$* is coNP-complete for ADFs.

3.3 Strong Justification of an Argument

Note that it is possible that an interpretation v contains some strongly justified arguments but v is not strongly admissible itself. Example 7 presents such an interpretation. Thus, the problem *StrJust*$(a \mapsto \mathbf{t}/\mathbf{f}, v, D)$ of deciding whether an argument is strongly justified in a given interpretation of an ADF is different from the previously discussed decision problems. We show that *StrJust* is coNP-complete.

Example 7. Let $D = (\{a, b, c, d\}, \{\varphi_a : \bot, \varphi_b : \neg a \wedge c, \varphi_c : d, \varphi_d : \top\})$ be an ADF. Let $v = \{b, c, d\}$ be an interpretation of D. It is easy to check that c and d are strongly acceptable in v. However, b is not strongly acceptable in D. Thus, v is not a strongly admissible interpretation of D. However, there exists a strongly admissible interpretation of D in which c and d are strongly acceptable and that has less information than v, namely, $v' = \{c, d\}$.

As discussed in Section 2.4, (Keshavarzi Zafarghandi, Verbrugge, and Verheij 2021b) presents a straightforward method of deciding whether a is strongly justified in a given interpretation v. That is, a is strongly acceptable/deniable in v if it is acceptable/deniable by the least fixed point of the operator $\Gamma_{D,v}$ (which is equal to $\Gamma^n_{D,v}(v_\mathbf{u})$ for sufficiently large n).

However, the repeated evaluation of Γ_D is a costly part of this algorithm and results in a P^{NP} algorithm. We will next discuss a more efficient method to answer this reasoning task. To this end, we translate a given ADF D to ADF D', presented in Definition 12, such that the queried argument is strongly justifiable in a given interpretation of D if and only if it is credulously justifiable in the grounded interpretation of D'. As shown in Proposition 4.1.3 in (Wallner 2014), the credulous decision problem for ADFs under grounded semantics is a coNP-problem. Thus, verifying whether a given argument is strongly justified in an interpretation is a coNP-problem, since the translation can be done in polynomial time with respect to the size of D.

Definition 12. Let $D = (A, L, C)$ be an ADF and let v be an interpretation of D. The translation of D under v is $D' = (A', L', C')$ such that $A' = A \cup \{x, y\}$ where $x, y \notin A$. Furthermore, for each $a \in A'$ we define the acceptance condition of a in D', namely φ'_a as follows:

- $\varphi'_x : x$;
- $\varphi'_y : y$;
- if $v(a) = \mathbf{u}$, then $\varphi'_a : \neg a$;
- if $v(a) = \mathbf{t}$, then $\varphi'_a = \varphi_a \vee x$;
- if $v(a) = \mathbf{f}$, then $\varphi'_a = \varphi_a \wedge y$.

Notice that our reduction ensures that arguments with $v(a) = \mathbf{u}$ will always be $\mathbf{u}$ in D', arguments with $v(a) = \mathbf{t}$ will be assigned to either $\mathbf{t}$ or $\mathbf{u}$ during the least fixed-point computation and arguments with $v(a) = \mathbf{f}$ will be assigned to either $\mathbf{f}$ or $\mathbf{u}$. That is we introduced arguments x, y to ensure that arguments in v^* are not assigned to the opposite truth value during the iteration of $\Gamma_{D'}$ that leads to the grounded interpretation of D'.

Lemmas 6 and 7 show the correctness of the reduction.

Lemma 6. *Let D be an ADF, let v be an interpretation of D, and let D' be the translation of D, via Definition 12. It holds that if StrJust$(a \mapsto \mathbf{t}/\mathbf{f}, v, D) = yes$, then Cred$_{grd}(a \mapsto \mathbf{t}/\mathbf{f}, D') = yes$.*

Proof. We assume that *StrJust*$(a \mapsto \mathbf{t}, v, D)$ = yes, and we show that *Cred*$_{grd}(a \mapsto \mathbf{t}, D')$ = yes. The proof for the case that *StrJust*$(a \mapsto \mathbf{f}, v, D)$ = yes is similar.

Assume that $v_\mathbf{u}$ is the trivial interpretation of D and $v'_\mathbf{u}$ is the trivial interpretation of D', i.e., $v'_\mathbf{u} = v_\mathbf{u} \cup \{x \mapsto \mathbf{u}, y \mapsto \mathbf{u}\}$. Assume that $\Gamma^i_{D,v}(v_\mathbf{u})$ is a sequence of strongly admissible interpretations constructed based on v in D, as in Definition 10. Let w be the limit of the sequence of $\Gamma^i_{D,v}(v_\mathbf{u})$.

StrJust$(a \mapsto \mathbf{t}, v, D) = yes$ implies that $w(a) = \mathbf{t}$. Since w is a strongly admissible interpretation of D, it holds that $a \mapsto \mathbf{t}$ in the grounded interpretation of D, i.e., there exists a natural number n such that $\Gamma^n_D(v_\mathbf{u})(a) = \mathbf{t}$. By induction on n, it is easy to show that $\Gamma^n_{D'}(v'_\mathbf{u})(a) = \mathbf{t}$. That is, a is assigned to $\mathbf{t}$ in the grounded interpretation of D'. Thus, *Cred*$_{grd}(a \mapsto \mathbf{t}, D')$ = yes. □

Lemma 7. *Let D be an ADF, let v be an interpretation of D, and let D' be the translation of D via Definition 12. It holds that if Cred$_{grd}(a \mapsto \mathbf{t}/\mathbf{f}, D) = yes$, then StrJust$(a \mapsto \mathbf{t}/\mathbf{f}, v, D) = yes$.*

Proof. Assume that a is justified in the grounded interpretation of D', namely w. Thus, there exists a j such that $w = \Gamma^j_{D'}(w_\mathbf{u})$ for $j \geq 0$, where $w_\mathbf{u}$ is the trivial interpretation of D'. By induction we prove the claim that for all i, if $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^i_{D'}(w_\mathbf{u})$, then a is strongly justified in v.

Base case: Assume that $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^1_{D'}(w_\mathbf{u})$. By the acceptance conditions of x and y in D', both of them are assigned to $\mathbf{u}$ in w. Then it has to be the case that either $\varphi'_a = \varphi_a \vee x$ or $\varphi'_a = \varphi_a \wedge y$ in D'. Thus, $a \mapsto \mathbf{t}/\mathbf{f} \in$

$\Gamma^1_{D'}(w_\mathbf{u})$ implies that ${\varphi'_a}^{w_\mathbf{u}} \equiv \top/\bot$. Thus, $w(x/y) = \mathbf{u}$, $\varphi'_a = \varphi_a \vee x/\varphi_a \wedge y$ and ${\varphi'_a}^{w_\mathbf{u}} \equiv \top/\bot$ together imply that ${\varphi_a}^{w_\mathbf{u}} \equiv \top/\bot$. Hence, $\varphi_a^{v_\mathbf{u}} \equiv \top/\bot$ where $v_\mathbf{u}$ is the trivial interpretation of D. That is, a is strongly justified in v.

Induction hypothesis: Assume that for all j with $1 \leq j \leq i$, if $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^j_{D'}(w_\mathbf{u})$, then a is strongly justified in v.

Inductive step: We show that if $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^{i+1}_{D'}(w_\mathbf{u})$, then a is strongly justified in v. Because $x/y \mapsto \mathbf{u} \in w$, we have that $\varphi_a^w \equiv \top/\bot$ implies that $\varphi_a^v \equiv \top/\bot$. Further, $a \mapsto \mathbf{t}/\mathbf{f} \in \Gamma^{i+1}_{D'}(w_\mathbf{u})$ says that there exists a set of parents of a, namely P, where $P \subseteq w^\mathbf{t} \cup w^\mathbf{f}$, such that, $\varphi_a^{w_{|P}} \equiv \top/\bot$. Thus, $\varphi_a^{v_{|P}} \equiv \top/\bot$. By induction hypothesis, each $p \in P$ is strongly justified in v. Thus, a is strongly justified in v. ☐

Theorem 5 is a direct result of Lemmas 6 and 7.

Theorem 5. Let D be an ADF, let v be an interpretation of D, and let D' be the translation of D, via Definition 12. It holds that *Cred*$_{grd}(a \mapsto \mathbf{t}/\mathbf{f}, D) = yes$ iff *StrJust*$(a \mapsto \mathbf{t}/\mathbf{f}, v, D) = yes$.

We use the auxiliary Theorem 5 to present the main result of this section, i.e., to show that *StrJust* is coNP-complete.

Lemma 8. *Let D be an ADF, let a be an argument, and let v be an interpretation of D. Deciding whether a is strongly justified in v, i.e., whether StrJust$(a \mapsto \mathbf{t}/\mathbf{f}, v, D)$, is a* coNP*-problem.*

Proof. It is shown in (Wallner 2014, Proposition 4.1.3) that the credulous decision problem under grounded semantics, i.e., *Cred*$_{grd}$, is a coNP-problem. Further, the translation of a given ADF D to D' via Definition 12 can be done in polynomial time. By Theorem 5, it holds that *Cred*$_{grd}(a \mapsto \mathbf{t}/\mathbf{f}, D) = yes$ iff *StrJust*$(a \mapsto \mathbf{t}/\mathbf{f}, v, D) = yes$. Thus, deciding whether a given argument is strongly justified in interpretation v, i.e., *StrJust*$(a \mapsto \mathbf{t}/\mathbf{f}, v, D)$ is a coNP-problem. ☐

Lemma 9. *Let D be an ADF, let a be an argument, and let v be an interpretation of D. Deciding whether a is strongly justified in v, i.e., StrJust$(a \mapsto \mathbf{t}/\mathbf{f}, v, D)$, is* coNP*-hard.*

Proof. Let ψ be any Boolean formula and let $X = atom(\psi)$ be the set of atoms in ψ. Let a be a new variable that does not appear in X. Construct $D = (\{X \cup \{a\}\}, L, C)$, such that $\varphi_x : x$ for each $x \in X$ and $\varphi_a : \psi$. ADF D can be constructed in polynomial time with respect to the size of ψ. We show that a is strongly acceptable in any v where $v(a) = \mathbf{t}$ if and only if ψ is a valid formula. An illustration of the reduction for a formula $\psi = \neg b \vee b$ to the ADF $D = (\{a, b\}, L, \varphi_a : \psi, \varphi_b : b)$ is depicted in Figure 3.

Assume that a is strongly acceptable in v, thus by Definition 8, there exists a set of parents of a, namely P, such that $\varphi_a^{v_{|P}} \equiv \top$ and for each $p \in P$ it holds that p is strongly justified in v. By the definition of D the acceptance condition of each parent of a, namely p is $\varphi_p : p$, thus, by the acceptance condition of p, it is not strongly justifiable in v. Thus, the only case in which a is strongly acceptable in v is that $P = \emptyset$, i.e., $\varphi_a^{v_\mathbf{u}} \equiv \top$. Hence, for any two-valued interpretation u of $X \cup \{a\}$ it holds that $u \models \psi$. Moreover since the atom a does not appear in ψ we obtain that for any two-valued interpretation u of X it holds that $u \models \psi$. Hence, ψ is a valid formula and it is a *yes* instance of the VALIDITY problem of classical logic.

On the other hand, assume that ψ is a valid formula. Then it is clear that the interpretation v that assigns a to $\mathbf{t}$ and x to $\mathbf{u}$, for each $x \in X$, is the grounded interpretation of D. Thus, the answer to the strong acceptance problem of a in any v with $v(a) = \mathbf{t}$ is *yes*.

For credulous denial of a, it is enough to present the acceptance condition of a equal to the negation of ψ in D, i.e., $\varphi_a : \neg\psi$, and follow a similar method. That is, a is strongly deniable in v, where $v(a) = \mathbf{f}$, if and only if ψ is a valid formula. ☐

Theorem 6 is a direct result of Lemmas 8 and 9.

Theorem 6. Let D be an ADF, let a be an argument, and let v be an interpretation of D. Deciding whether a is strongly justified in v, i.e., *StrJust*$(a \mapsto \mathbf{t}/\mathbf{f}, v, D)$ is coNP-complete.

3.4 Smallest Witness of Strong Justification

Assume that an argument a, its truth value, and a natural number k are given. We are eager to know whether there exists a strongly admissible interpretation v that satisfies the truth value of a and $|v^\mathbf{t} \cup v^\mathbf{f}| < k$. This reasoning task is denoted by *k-Witness*$_{sadm}(a \mapsto \mathbf{t}/\mathbf{f}, D)$. We show that *k-Witness*$_{sadm}$ is Σ_2^P-complete. Lemma 10 shows that this problem is a Σ_2^P-problem and Lemma 11 indicates the hardness of this reasoning task.

Lemma 10. *Let $D = (A, L, C)$ be an ADF, let a be an argument, let $x \in \{\mathbf{t}, \mathbf{f}\}$, and let k be a natural number. Deciding whether there exists a strongly admissible interpretation v of D where $v(a) = x$ and $|v^\mathbf{t} \cup v^\mathbf{f}| < k$ is a Σ_2^P-problem, i.e., k-Witness$_{sadm}$ is a Σ_2^P-problem.*

Proof. For membership, non-deterministically guess an interpretation v and verify whether this interpretation satisfies the following items:

1. $v \in sadm(D)$;
2. $v(a) = x$;
3. $|v^\mathbf{t} \cup v^\mathbf{f}| < k$.

If v satisfies all the items, then the answer to the decision problem is *yes*, i.e., *k-Witness*$_{sadm}(a \mapsto \mathbf{t}/\mathbf{f}, D) = yes$. The complexity of each of the above items is as follows.

1. Verifying strong admissibility of v is coNP-complete, as is presented in Section 3.2.
2. Verifying if v contains the claim, i.e., if $v(a) = x$, can clearly be done in polynomial time.
3. Collecting $v^\mathbf{t} \cup v^\mathbf{f}$ and checking whether $|v^\mathbf{t} \cup v^\mathbf{f}| < k$ takes only polynomial time.

That is, the algorithm first non-deterministically guesses an interpretation v and then performs checks that are in coNP to verify that v satisfies the requirements of the decision problem. Thus, this gives an $\mathsf{NP}^{\mathsf{coNP}} = \Sigma_2^\mathsf{P}$ procedure. ☐

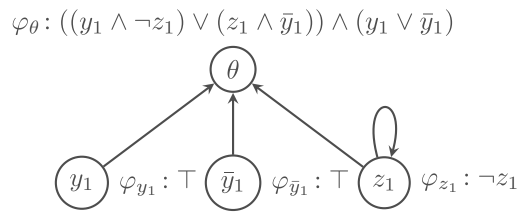

Figure 4: Illustration of the reduction from the proof of Lemma 11 for $\Theta = \exists y_1 \forall z_1((y_1 \wedge \neg z_1) \vee (z_1 \wedge \neg y_1)) \wedge (y_1 \vee \neg y_1)$.

Lemma 11. *Let $D = (A, L, C)$ be an ADF, let a be an argument, let $x \in \{\mathbf{t}, \mathbf{f}\}$, and let k be a natural number. Deciding whether there exists a strongly admissible interpretation v of D where $v(a) = x$ and $|v^{\mathbf{t}} \cup v^{\mathbf{f}}| < k$ is Σ_2^{P}-hard, i.e., k-Witness$_{sadm}$ is Σ_2^{P}-hard.*

Proof. Consider the following well-known problem on quantified Boolean formulas. Given a formula $\Theta = \exists Y \forall Z\, \theta(Y, Z)$ with atoms $X = Y \cup Z$ (and $Y \cap Z = \emptyset$) and propositional formula θ. Deciding whether Θ is valid is Σ_2^{P}-complete (see e.g. (Arora and Barak 2009)). We can assume that θ is of the form $\psi \wedge \bigwedge_{y \in Y}(y \vee \neg y)$, where ψ is an arbitrary propositional formula over atoms X, and that θ is satisfiable. Moreover, we can assume that the formula θ only uses $\wedge, \vee, \neg$ operations and negations only appear in literals. Let $\bar{Y} = \{\bar{y} : y \in Y\}$, i.e., for each $y \in Y$ we introduce a new argument $\bar{y}$.

We construct an ADF $D_\Theta = (A, L, C)$ with

$$A = Y \cup \bar{Y} \cup Z \cup \{\theta\}$$
$$C = \{\varphi_y : \top \mid y \in Y\} \cup \{\varphi_{\bar{y}} : \top \mid y \in Y\} \cup \{\varphi_z : \neg z \mid z \in Z\} \cup \{\varphi_\theta : \theta[\neg y/\bar{y}]\}$$

It is easy to verify that the grounded interpretation g of D_Θ sets all arguments $Y \cup \bar{Y}$ to $\mathbf{t}$ and all arguments Z to $\mathbf{u}$. Moreover, $g(\theta) \in \{\mathbf{t}, \mathbf{u}\}$. An illustration of the reduction for a formula $\theta = ((y_1 \wedge \neg z_1) \vee (z_1 \wedge \neg y_1)) \wedge (y_1 \vee \neg y_1)$ to the ADF $D = (A, L, C)$ is shown in Figure 4, where: $A = \{y_1, \bar{y_1}, z_1, \theta\}$, $\varphi_{y_1} : \top, \varphi_{\bar{y}_1} : \top, \varphi_{z_1} : \neg z$ and $\varphi_\theta : ((y_1 \wedge \neg z_1) \vee (z_1 \wedge \bar{y}_1)) \wedge (y_1 \vee \bar{y}_1)$. We show that there is a strongly admissible interpretation v with $v(\theta) = \mathbf{t}$ and $|S| = |Y| + 1$ where $S = v^{\mathbf{t}} \cup v^{\mathbf{f}}$ iff Θ is a valid formula.

- Assume that Θ is a valid formula. We show that there exists a strongly admissible interpretation v with $|S| = |Y| + 1$. Since Θ is a valid formula, there exists an interpretation I_Y of Y such that for any interpretation I_Z of Z, it holds that $I_Y \cup I_Z \models \theta(Y, Z)$, i.e., θ is true. Specifically, it holds that $I_Y \models \theta(Y, Z)$.
  We define a three-valued interpretation v of A such that $v(y) = \mathbf{t}$ if $I_Y(y) = \mathbf{t}$, $v(\bar{y}) = \mathbf{t}$ if $I_Y(y) = \mathbf{f}$, $v(\theta) = \mathbf{t}$, and $v(x) = \mathbf{u}$ otherwise. It is easy to check that v is a strongly admissible interpretation of D where $|S| = |Y| + 1$. Thus, θ is strongly acceptable in a strongly admissible interpretation v where $|S| = |Y| + 1$.
- Let v be the strongly admissible interpretation with $v(\theta) = \mathbf{t}$ and $|S| \leq |Y| + 1$. Let g be the unique grounded interpretation of D. It holds that $v \leq_i g$. For each $z \in Z$, since $c_z : \neg z$, it is clear that $v(z) = \mathbf{u}$ in any strongly admissible interpretation v of D. Moreover, because θ is of the form $\psi \wedge \bigwedge_{y \in Y}(y \vee \neg y)[\neg y/\bar{y}]$, we have that for each $y \in Y$ either $v(y) = \mathbf{t}$ or $v(\bar{y}) = \mathbf{t}$ and thus $|S| = |Y| + 1$. Because of this, we also have that not both $v(y) = \mathbf{t}$ or $v(\bar{y}) = \mathbf{t}$ can be simultaneously true. We can thus define the following interpretation I_Y of Y such that $I_Y(y) = \mathbf{t}$ if $v(y) = \mathbf{t}$ and $I_Y(y) = \mathbf{f}$ if $v(\bar{y}) = \mathbf{t}$. Since θ is strongly accepted with respect to v, we have that for each interpretation I_Z of Z, the formula θ is satisfied by $I_Y \cup I_Z$. That is, the QBF Θ is valid.

□

	Cred$_{sadm}$	*Skept$_{sadm}$*	*Ver$_{sadm}$*	*StrJust*	*k-Witness$_{sadm}$*
AFs	P	trivial	P	n.a.	NP-c
ADFs	coNP-c	trivial	coNP-c	coNP-c	Σ_2^{P}-c

Table 1: Complexity under the strong admissibility semantics of AFs and ADFs ($\mathcal{C}$-c denotes completeness for class $\mathcal{C}$)

Theorem 7 is a direct result of Lemmas 10 and 11.

Theorem 7. *k-Witness$_{sadm}$* is Σ_2^{P}-complete.

In Table 1, we summarize our results on the complexity of strong admissibility semantics in ADFs and compare them with the corresponding results for AFs (Caminada and Dunne 2020; Dvořák and Wallner 2020).

4 Conclusion

We studied the computational properties of the strong admissibility semantics of ADFs. When compared to AFs, computational complexity for ADFs increases by one step in the polynomial hierarchy (Stockmeyer 1976) for nearly all reasoning tasks (Strass and Wallner 2015; Dvořák and Dunne 2018). We have shown that, similarly, ADFs have higher computational complexity under the strong admissibility semantics when compared to AFs (Table 1).

From a theoretical perspective we observe that: 1. The credulous decision problem under the strong admissibility semantics of ADFs is coNP-complete, while this decision problem is tractable in AFs. 2. Since the trivial interpretation is the least strongly admissible interpretation for each ADF, the skeptical decision problem is trivial, which is similar for AFs. 3. The verification problem for ADFs is coNP-complete, while it is tractable for AFs. 4. Since an argument can be strongly justified in an interpretation that is not a strongly admissible interpretation, we defined a new reasoning task in Section 3.3, called the strong justification problem. The complexity of this decision problem, which investigates whether a queried argument is strongly justified in a given interpretation, is coNP-complete. 5. The problem of finding a smallest witness of strong justification of an argument investigates whether there exists a strongly admissible interpretation that assigns a minimum number of arguments to $\mathbf{t}/\mathbf{f}$ and satisfies a queried argument is Σ_2^{P}-complete, while this reasoning task is NP-complete for AFs.

We next highlight an interesting difference in the complexity landscapes of AFs and ADFs. When relating the complexity of grounded and strong admissibility semantics, we have that for AFs the verification problems can be (log-space) reduced to each other, while for ADFs there is a gap between the coNP-complete Ver_{sadm} problem and the DP-complete Ver_{grd} problem. That is, on the ADF level the step of proving arguments to be $\mathbf{u}$ in the grounded interpretation adds an NP part to the complexity; a similar effect can be observed for admissible and complete semantics.

As future work, it would be interesting to analyse the computational complexity of the current reasoning tasks for strong admissibility semantics over subclasses of ADFs, in particular bipolar ADFs (Brewka and Woltran 2010) and acyclic ADFs (Diller et al. 2020).

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